Big Ramsey degrees, structures, dynamics II

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Outline of part II.

- Minimal subflows of $\mathrm{Sa}(G)$.

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- Extreme amenability and KPT correspondence.
- Metrizable universal minimal flows.

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- $\mathrm{Sa}(G)$, the Samuel compactification of G, is the set of near ultrafilters on G, maximal $\mathsf{p} \subseteq \mathcal{P}(G)$ with the property that for any $F \in [\mathsf{p}]^{<\omega}$ and $U \in \mathcal{N}_G$, we have $\bigcap_{S\in F} SU \neq \emptyset$.

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- We set $N_S = \{\mathsf{p} \in \mathrm{Sa}(G) : S \notin \mathsf{p}\}$ and $C_S = \{\mathsf{p} \in \mathrm{Sa}(G) : S \in \mathsf{p}\}$. The topology on $\mathrm{Sa}(G)$ given by basis $\{N_S : S \subseteq G \text{ not dense}\}$ is compact Hausdorff.

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- We set $N_S = \{\mathsf{p} \in \mathrm{Sa}(G) : S \notin \mathsf{p}\}$ and $C_S = \{\mathsf{p} \in \mathrm{Sa}(G) : S \in \mathsf{p}\}$. The topology on $\mathrm{Sa}(G)$ given by basis $\{N_S : S \subseteq G \text{ not dense}\}$ is compact Hausdorff.
- G acts on $\mathrm{Sa}(G)$ in the natural way. Any minimal subflow of $\mathrm{Sa}(G)$ is isomorphic to the universal minimal flow $\mathrm{M}(G)$.

Let's try to describe minimal subflows of $\mathrm{Sa}(G)$ in near ultrafilter language. First, what about closed subsets? In $\beta\mathbb{N}$, 1-1 correspondence between closed subsets and filters on $\mathbb{N}$...

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Fact (Exercise 1)

There is a 1-1 correspondence between closed subsets of $\mathrm{Sa}(G)$ and near filters on G.

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Fact (Exercise 2)

$C_S \subseteq \mathrm{Sa}(G)$ contains a subflow iff $S \subseteq G$ is pre-thick. In particular, S is pre-thick iff the collection $\{Sg : g \in G\}$ has the near FIP.

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Proof sketch.

If Q is a near filter and some $S \in \mathsf{Q}$ is not pre-thick, then C_S does not contain a subflow, so also $\bigcap_{S \in \mathsf{Q}} C_S$ cannot contain a subflow.

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If Q is a near filter all of whose members are pre-thick, first note that $\bigcap_{S \in \mathsf{Q}} C_S = \bigcap_{S \in \mathsf{Q}, U \in \mathcal{N}_G} C_{SU}$. Right hand side gives a directed intersection of compact sets containing subflows. □

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We say that $\mathbf{A}$ is a Ramsey object if for every $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and $0 < r < \omega$, there is $\mathbf{B} \leq \mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_r$. $\mathcal{K}$ has the Ramsey property if every $\mathbf{A} \in \mathcal{K}$ is a Ramsey object.

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Fact (Exercise 3)

In the definition of RP, equivalent to only consider $r = 2$.

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- finite anti-lexicographically ordered Boolean algebras (not relational) (Graham-Rothschild 1971)

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Fact (Exercise 4)

$\mathbf{A} \in \mathcal{K}$ is a Ramsey object iff for every $\mathbf{B} \in \mathcal{K}$, $0 < r < \omega$, thick $T \subseteq \mathrm{Emb}_{\mathbf{A}}$, and coloring $\gamma\colon T \to r$, there is $x \in \mathrm{Emb}_{\mathbf{B}}$ with $|\{\gamma(x \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| = 1$.

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Hence $\mathbf{A} \in \mathbf{K}$ is a Ramsey object iff the collection $\mathcal{T}_{\mathbf{A}}$ of thick subsets of $\mathrm{Emb}_{\mathbf{A}}$ is a coideal.

Now set $G = \mathrm{Aut}(\mathbf{K})$. Recall that given $\mathbf{A} \in [\mathbf{K}]^{<\omega}$, we write $U_{\mathbf{A}} = \mathrm{Stab}(\mathbf{A})$, and we identify $G/U_{\mathbf{A}}$ with $\mathrm{Emb}_{\mathbf{A}}$

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Theorem (Kechris-Pestov-Todorčević 2005)

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Of course, this happens exactly when $\mathrm{Sa}(G)$ has a fixed point...

Proof sketch (Z. 2016).

Suppose $\mathcal{K}$ has the Ramsey Property. Fix finite $\mathbf{A}_0 \subseteq \mathbf{A}_1 \subseteq \cdots$ with $\bigcup_{n<\omega} \mathbf{A}_n = \mathbf{K}$. Write Emb_n, U_n, $\mathcal{T}_n$, etc. RP tells us that each $\mathcal{T}_n$ is a coideal.

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Inductively define $p_n \in \beta\mathrm{Emb}_n$ as follows. Let $p_0 \subseteq \mathcal{T}_0$ be any ultrafilter.

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If thick ultrafilter p_n has been defined, first form the filter $q_{n+1} = \langle\{x \in \mathrm{Emb}_{n+1} : x|_{\mathbf{A}_n} \in S\} : S \in p_n\rangle$. Then $q_{n+1} \subseteq \mathcal{T}_{n+1}$ is a thick filter, and extends to an ultrafilter $p_{n+1} \subseteq \mathcal{T}_{n+1}$.

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Define $\mathsf{p} = \{S \subseteq G : \forall n < \omega\, (SU_n \in p_n)\}$. Then $\mathsf{p} \in \mathrm{Sa}(G)$ and consists entirely of pre-thick sets. □

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Fact (Exercise 5)

If $\mathcal{K}$ does not have RP, then for some $\mathbf{A} \in \mathcal{K}$, there is a syndetic 2-coloring of $\mathrm{Emb}_{\mathbf{A}}$. If $\gamma \in 2^{\mathrm{Emb}_{\mathbf{A}}}$ is a syndetic coloring and G acts on $2^{\mathrm{Emb}_{\mathbf{A}}}$ via $(\delta \cdot g)(f) = \delta(g \circ f)$, then $\overline{\gamma \cdot G}$ is a G-flow with no fixed points.

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Definition

$\mathbf{A} \in \mathcal{K}$ has finite small Ramsey degree if there is $t_{\mathbf{A}} < \omega$ such that for any $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$, $0 < r < \omega$, and coloring $\gamma\colon \mathrm{Emb}_{\mathbf{A}} \to r$, there is $x \in \mathrm{Emb}_{\mathbf{B}}$ with $|\{\gamma(x \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| \leq t_{\mathbf{A}}$. The least such $t_{\mathbf{A}}$ is called the small Ramsey degree (SRD) of $\mathbf{A}$.

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The definition can be completely finitized, but we go the other way: $\mathbf{A}$ has SRD $t_{\mathbf{A}}$ iff for any $0 < r < \omega$ and coloring $\gamma\colon \mathrm{Emb}_{\mathbf{A}} \to r$, there is $I \subseteq r$ with $|I| \leq t_{\mathbf{A}}$ and $\gamma^{-1}(I)$ thick.

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In particular, there is a thick filter on $\mathrm{Emb}_{\mathbf{A}}$ corresponding to a finite closed subset of $\beta\mathrm{Emb}_{\mathbf{A}}$ of size exactly $t_{\mathbf{A}}$.

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(3) $\Rightarrow$ (1) appears in KPT. (1) $\Rightarrow$ (2) follows from considerations on the previous slide. For (2) $\Rightarrow$ (3) we present a variant of a simpler proof due to Nguyen Van Thé.

Definition

Given $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and finite colorings $\gamma_{\mathbf{A}}, \gamma_{\mathbf{B}}$ of $\mathrm{Emb}_{\mathbf{A}}, \mathrm{Emb}_{\mathbf{B}}$, respectively, we say $\gamma_{\mathbf{A}} \ll \gamma_{\mathbf{B}}$ iff whenever $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ and $x, y \in \mathrm{Emb}_{\mathbf{B}}$ satisfy $\gamma_{\mathbf{B}}(x) = \gamma_{\mathbf{B}}(y)$, then $\gamma_{\mathbf{A}}(x \circ f) = \gamma_{\mathbf{B}}(y \circ f)$.

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A rephrase of Ramsey degree: $\mathbf{A}$ has Ramsey degree $t_{\mathbf{A}}$ if this is least so that for any finite coloring γ of $\mathrm{Emb}_{\mathbf{A}}$, there is $\gamma' \in \overline{\gamma \cdot G}$ which takes at most $t_{\mathbf{A}}$ values.

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In particular, if $\mathbf{A} \in \mathcal{K}$ has small Ramsey degree $t_{\mathbf{A}}$, then there is a syndetic $t_{\mathbf{A}}$-coloring of $\mathrm{Emb}_{\mathbf{A}}$.

Fact ((2) ⇒ (3) of theorem)

If each $\mathbf{A} \in \mathcal{K}$ *has finite small Ramsey degree* $t_{\mathbf{A}}$*, then there are* $\{\gamma_{\mathbf{A}} : \mathbf{A} \in [\mathbf{K}]^{<\omega}\}$ *with each* $\gamma_{\mathbf{A}}$ *a syndetic* $t_{\mathbf{A}}$*-coloring and with* $\gamma_{\mathbf{A}} \ll \gamma_{\mathbf{B}}$ *whenever* $\mathbf{A} \leq \mathbf{B}$*.*

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Key idea: Any coloring in the orbit closure of a syndetic t-coloring is still a syndetic t-coloring.

Start with any collection $\{\gamma^0_{\mathbf{A}} : \mathbf{A} \in [\mathbf{K}]^{<\omega}\}$ of syndetic $t_{\mathbf{A}}$-colorings. Enumerate all pairs from $[\mathbf{K}]^{<\omega}$ with $\mathbf{A} \leq \mathbf{B}$.

If colorings $\gamma^n_{\mathbf{A}}$ are determined, we now handle $\mathbf{A}_n \leq \mathbf{B}_n$. Let δ be the finite coloring of $\mathrm{Emb}_{\mathbf{B}_n}$ formed using all information from $\gamma^n_{\mathbf{B}_n}$ and $\gamma^n_{\mathbf{A}_n}$. Find a sequence $(g_k)_{k<\omega}$ from G such that both of the following:

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Note that δ' depends only on $\gamma^{n+1}_{\mathbf{B}_n}$. It follows that $\gamma^{n+1}_{\mathbf{B}_n} \ll \gamma^{n+1}_{\mathbf{A}_n}$.

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Let $(\gamma_{\mathbf{A}})_{\mathbf{A}\in[\mathbf{K}]^{<\omega}} \in \prod_{\mathbf{A}\in[\mathbf{K}]^{<\omega}} t_{\mathbf{A}}^{\mathrm{Emb}_{\mathbf{A}}}$ be any limit point of the sequence $(\gamma^n_{\mathbf{A}})_{\mathbf{A}\in[\mathbf{K}]^{<\omega}}$. Each $\gamma_{\mathbf{A}}$ is in $\overline{\gamma^0_{\mathbf{A}} \cdot G}$, so is $t_{\mathbf{A}}$-syndetic. As $\ll$ is a closed condition, we get the result.